LECTURE T5 – CO-ORDINATE CALCULATIONS **PRESENTER: CD SCOTT**

1 APPLICATION

Paper is a patient medium, and it is possible to create the most complex combinations of straight lines and curves of constantly varying radii. For example, designers, at one stage, were very taken with the concept of curvilinear alignment. This was a radical departure from the traditional system of locating tangents in the first instance and then connecting these with curves. The curvilinear approach was considered frightfully artistic and principally comprised the initial location of curves - circular, compound or spiral - and then connecting these with relatively short straights. The end result may have been visually pleasing, but it certainly didn't do the driver any favours and also presented the road builder with a challenge that he could quite comfortably have lived without. In terms of our philosophy of systems analysis it is an example of pursuing a minor objective by sacrificing a major objective - a contradiction in terms.

The designer, in developing his design of an abstract ribbon in space, needs to define the shape he has created in some fashion, or it otherwise remains nothing more than a pretty picture. The system of definition universally employed is to describe a series of points along the route. The route then comprises the line passing through these points. Any point in space can be defined by the use of three numbers, referred to as the co-ordinates of the point, with each number representing a distance from a given datum point. The designer could simply scale these distances off his drawing in respect of each point. He would however have some difficulty in describing his datum point to the road builder. Furthermore, scaling to an accuracy of one millimetre on a drawing may demonstrate that a straight line is distinctly crooked when subjected to scaling up by a factor of 1 000. A greater degree of precision than that provided by scaling is required. This is provided by co-ordinate calculation.

Ultimately, the picture that the designer has created has to be transferred from the drawing board into reality - the route must be set out in other words. This represents one of the restraints under which the designer works because, if it is not possible to set out the design, all his efforts have been a waste of time. Setting out comprises the process of relating the points on his route as defined by the designer to known points on the earth's surface. These known points comprise a hierarchy of primary, secondary, and tertiary trigonometrical survey beacons, located to high orders of accuracy. The surveyor uses these to establish points fairly close to where the road is ultimately meant to be located and then measures from these local survey beacons either to centreline pegs or to reference pegs offset to one side of the road or the other. The surveyor is thus also concerned with co-ordinate calculation. With rapid advancements in the accuracy and availability of survey instruments that make use of satellite tracking and positioning, it is common practice today to use this form of surveying for most road design and construction projects. The fundamental differences of this form of survey to that previously adopted is covered in more detail in section 2 hereafter.

2 THE GRID SYSTEM

Prior to the use of satellite positioning, each country or geographical area around the world established and used their own system of survey measurement. In South Africa, the datum point from which all measurements on the earth's surface are made is the intersection of Longitude 0, which passes through Greenwich, and the Equator. The system used in the Northern Hemisphere is known as the Gauss System. True North is taken as the zero bearing and angles are measured in an anti-clockwise direction. Distances increase from South to North on the x-axis and from West to East on the y-axis.

In the Southern Hemisphere, the Gauss System is also used but the system is rotated twice. Rotation is about the Equator and then about Lo 0. The zero bearing is thus true South. Distances increase from North to South on the x-axis and from East to West on the y-axis. Confusingly enough, when distances are quoted in terms of degrees longitude, the positive direction is still from West to East. Bearings increase in a clockwise direction. This rotated system is known as the Gauss Conform System.

Elevations are typically measured from a datum of mean sea level (MSL) or low water ordinary spring tide (LWOST).

The more commonly used system these days is the Universal Transverse Mercator projection and grid system which was adopted by the U.S. Army in 1947 for designating rectangular coordinates on large scale military maps. With the advent of inexpensive GPS receivers, many other map users are adopting the UTM grid system for coordinates that are simpler to use than latitude and longitude.

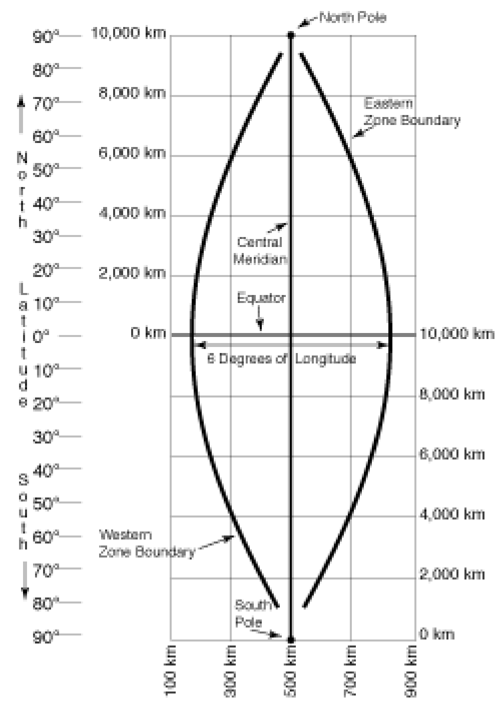
The UTM system divides the earth into 60 zones each 6 degrees of longitude wide. These zones define the reference point for UTM grid coordinates within the zone. UTM zones extend from a latitude of 80° S to 84° N. In the polar regions the Universal Polar Stereographic (UPS) grid system is used.

UTM zones are numbered 1 through 60, starting at the international date line, longitude 180°, and proceeding east. Zone 1 extends from 180° W to 174° W and is centred on 177° W.

Each zone is divided into horizontal bands spanning 8 degrees of latitude. These bands are lettered, south to north, beginning at 80° S with the letter C and ending with the letter X at 84° N. The letters I and O are skipped to avoid confusion with the numbers one and zero. The band lettered X spans 12° of latitude.

A square grid is superimposed on each zone. It's aligned so that vertical grid lines are parallel to the centre of the zone, called the central meridian.

UTM grid coordinates are expressed as a distance in metres to the east, referred to as the "easting", and a distance in metres to the north, referred to as the "northing".



Eastings

UTM easting coordinates are referenced to the centre line of the zone known as the central meridian. The central meridian is assigned an easting value of 500,000 metres East. Since this 500,000m value is arbitrarily assigned, eastings are sometimes referred to as "false eastings"

An easting of zero will never occur, since a 6° wide zone is never more than 674,000 metres wide.

Minimum and maximum easting values are:

160,000 mE and 834,000 mE at the equator

465,000 mE and 515,000 mE at 84° N

Northings

UTM northing coordinates are measured relative to the equator. For locations north of the equator the equator is assigned the northing value of 0 metres North. To avoid negative numbers, locations south of the equator are made with the equator assigned a value of 10,000,000 metres North.

Some UTM northing values are valid both north and south of the equator. In order to avoid confusion, the full coordinate needs to specify if the location is north or south of the equator. Usually this is done by including the letter for the latitude band.

If this is your first exposure to the UTM coordinate system you may find the layout of zones to be confusing. In most land navigation situations, the area of interest is much smaller than a zone. The notion of a zone falls away and we are left with a simple rectangular coordinate system to use with our large-scale maps.

Frequently, in land navigation, the zone information and the digits representing 1,000,000m, and 100,000m are dropped. The 1m, 10m and 100m digits are used only to the extent of accuracy desired. Note that it's the smaller digits that are dropped in the notation used by the USGS on the edges of their maps. For example, 4282000 m N. becomes 82.

ALL UTM zones cover six-degree swaths. In South Africa, where we use Transverse Mercator as well as Universal Transverse Mercator (UTM), each of our Lo systems cover only two degree swaths. As a result of this difference, in SA we have a scale enlargement correction of 1.0, in other words there is no scale enlargement. In any other area where UTM is utilised there is a scale enlargement correction of 0.9996 – this is a mean scale enlargement correction over a six-degree swath. There will be discrepancies in scale enlargement over the six-degree swath but this is the mean and is applied throughout the zone. A more precise correction for a specific area can be achieved by measuring with a total station between any 2 points that have been fixed with the GPS and divide the total station distance by the GPS UTM obtained distance and this will give you a factor to apply to any further measurements in the area.

With the grid properly defined, any point on the Earth's surface can thus be completely described by a distance from each of the datum planes and is thus quoted as (y, x, z) with x being measured from the Equator, y from Lo 0 and z from MSL or LWOST. In practice, measuring from the Equator and Lo 0 would result in being required to use some rather unmanageably large numbers and the datum points are shifted by the use of constants. In the Western Cape, the y value is measured from Lo 19 and, by the time you get to the Eastern Free State, reference is to Lo 27. The datum for values of x is shifted by 10 000 000 m South from the Equator. The value of z is the height above sea level so that reference is sometimes to h instead of z. Just about all of South Africa is at an altitude of 2 000 m or less so that it is not necessary to have a shifted datum for altitude.

It should be self-evident that, whatever datum shift is selected, it must be common to all the points for which co-ordinate calculations are required. Trying to calculate a distance between two points when one is quoted on the basis of Lo 21 and the other on Lo 23 is downright inconvenient.

It does not automatically follow that this system is always employed. A survey of limited extent may perhaps be based on a purely arbitrary local grid. The surveyor confronted by an elderly farm diagram could well find that one of the corners is taken as being (0, 0) and one of the boundary lines taken as 0^0, with the unit of measurement being the Rhineland rood. Height could be measured from the top of a peg driven into the ground at some convenient point, with the top of the peg being assumed to be at a level of 100,00.

Local systems can represent a major problem to the surveyor. It is not known whether this system has now been scrapped but some 25 years ago, any work which went over the boundary between the Cape Town Municipal and Divisional Council areas had to take cognisance of the fact that the Divisional Council worked on Gauss Conform from Lo 19 and a datum of mean sea level whereas the Cape Town Local System had a zero direction which was not quite true South and a datum of Low Water Ordinary Spring Tide. The actual co-ordinates did not differ all that much so that if the surveyor did not know about the differences in system, he could spend many happy days looking for a mistake in his calculations.

The Rhineland rood is equal to 12 Cape feet which is the unit in which all horizontal distances were measured. Heights were always measured in English feet. This, of course, meant that many errors were made in the calculation of gradient, although design invariably used English feet throughout. Design now employs the metre, but there are still many farm diagrams on which the old units of measurement appear.

3 BASIC CALCULATIONS

All the designer requires is a reasonable knowledge of geometry and trigonometry and the ability to solve a single equation of the form

$$x_2 = x_1 + \delta x$$

A few other manipulations may also be called for, of course. For the purpose of this discussion, we are however limiting ourselves to the single equation. Solution can be either to find δx with known values of x_1 and x_2, or to find x_2 from known values of x_1 and δx. Almost all forms of co-ordinate calculation are merely various combinations of the two presentations of this equation known respectively as the Join and the Traverse (or Polar).

3.1 The Join

The only time the designer ever actually scales anything is when he determines the co-ordinates of the two points that are the terminals of a straight. These are the points of intersection (PI's) of the straight of interest with the straights on either side of it. The distance between the two points is required, if for no reason other than to determine chainages, also referred to as stake values in the metric system, along the route. The bearing calculated is required to be able to calculate curve data.

The major application of the Join is in the setting out of works. The surveyor sets up over a known point and requires to peg another point given to him by the designer's calculations. He therefore needs to know what distance to tape from his set up point and in what direction.

Setting out is normally done to an accuracy of 5 seconds of bearing and 10 mm of distance so that the scaled values are used to calculate a distance and bearing between the two points. These are then rounded off to the nearest 5 s and 10 mm and the coordinates of the second point recalculated. Scaling thus only applies really to the first point of the entire route. It invariably is the case that the terminals of the route are its intersections with other roads. If these roads' centrelines have been fixed by co-ordinates then the location of the terminals is also by calculation rather than scaling.

The calculation thus takes the form of getting the differences between the two sets of values (given as (y,x) with y increasing to the West and x increasing to the South) defining the points. The bearing is determined from

$$\tan \Theta \quad = \quad {}^{\delta y} / {}_{\delta x}$$

The distance can be calculated in one of three ways, one of which is recourse to the Theorem of Pythagoras. Normally, however the calculation used is

$$L \quad = \quad \delta y / \sin \Theta$$

or

$$L \quad = \quad \delta x / \cos \Theta$$

Surveyors are somewhat attached to self-checking calculations so that both are usually employed. The value of sine changes at its slowest in the region of 90^o and cosine at its slowest at 0^o, so that the preferred calculation is dependant on the value of Θ.

Presenting the calculation in tabular form it is thus

Item	**Y**	**X**
PI 1	y_1	x_1
PI 2	y_2	x_2
Difference δ	$\delta y = y_2 - y_1$	$\delta x = x_2 - x_1$
tan Θ	$\delta y/\delta x$	
Θ	$\tan^{-1} \delta y/\delta x$	
L	Δy / sin Θ	Δx / cos Θ

Most designers have programmable calculators, or alternatively the whole process of alignment calculation is computerised. However, the above tabulation demonstrates the process to be carried out and the programming effort required is, at most, modest.

3.2 The Polar (Traverse)

The Polar is the calculation used during traverses, hence its alternative name. The fieldwork involved in a traverse includes setting up at a point and reading the bearing and distance to the next point after having taken a back sight to the previous point for orientation purposes. In order to check the accuracy of the work, the traverse proceeds from a known point and closes on another known point. It may also close at its starting point, when it is known as a closed traverse.

The calculation required is to establish the co-ordinates of the various points along the traverse. One application of this process is the determination of the co-ordinates of survey beacons along a route for subsequent use in setting out operations.

It is quite likely that errors (not to be confused with mistakes) in distance or bearing measurements will result in the calculated co-ordinates of a point not matching its known values. An adjustment, usually in terms of Bowditch's Compass Rule, is then applied. This requires a join between the calculated and known values of the co-ordinates to quantify the

extent of the error. This is then removed by a series of polars, one at each measured point. The polar uses the initially calculated co-ordinates and the bearing of the total error.
The distance used is a fraction of the error distance derived and is proportional to the ratio between total length of traverse and the distance from start of traverse to the point of interest.

Presenting the calculation as a tabulation, it takes the form

Calculation Of The Polar

Item	Y	X	Ray
Co-ords of Pt 1	y_1	x_2	L
Difference	L sin Θ	L cos Θ	Θ
Co-ords of Pt 2	$y_2 = y_1 + L \sin \Theta$	$x_2 = x_1 + L \cos \Theta$	

4 SETTING OUT OF WORKS

The designer has toiled mightily on a 1:50 000 topographic map locating the route between Poepieskloof and Skilpadvrekvandors. Being a sensible man, he stayed in his air-conditioned office and sent the surveyor out on site. The surveyor put in a line of survey beacons and marked then with nice large white crosses, where after a run of aerial photographs is converted into a strip survey.

The designer once again performs his esoteric magic and produces a set of scaled coordinates of Points of Intersection (PI's) and the radius of curvature that he has in mind at each PI. The question is what is the surveyor now supposed to do? And what he does is the setting out of the road.

Inevitably, the scaled co-ordinates are going to produce bearings that run into the fourth decimal of a second and distances that are to the same order of millimetres. He will therefore go through a sequence of joins and polars commencing at the first tangent. The bearing and distance obtained from a join between the scaled co-ordinates are rounded off to something more useful and these are then used in a polar to calculate revised co-ordinate values of the next PI. The process is repeated for each tangent up to the last PI.

4.1 Setting out tangents

The difference between the first and the second bearing at the PI is referred to as *the angle of* *deviation*, Θ.

The tangent length of the curve is the distance between the PI and the start or end of curve (BC and EC) and this is

$$T = R \tan \Theta/2$$

The co-ordinates of the BC are derived by a polar using the PI coordinates, the tangent length, T, and the reciprocal of the bearing from PI 1 to PI 2 (the reciprocal simply being the bearing plus or minus 180^0). The EC coordinates are similarly derived by use of polar using the bearing from PI 2 to PI 3. By definition, the first and last PI's do not define the location of curves and are the starting and end points of tangents. This process results in a string of co-ordinate values between which the surveyor can set out the tangents, e.g. from EC 1 to BC 2.

The surveyor can now inspect his arsenal of survey beacons and select those closest to these points on the location of the route. For the purpose of this explanation, survey beacons A and B are fairly close to EC 5 and beacons C and D close to BC 6. A join from Beacon B to EC 5 will give the bearing and distance between beacon and EC.

The bearing doesn't help all that much until he knows which way is South. A join between the two survey beacons, A and B, is required and this provides a bearing between two points which are defined on the ground by steel pegs set in concrete. If he sets his tacheometer up over Beacon B and sights a ranging rod held at Beacon A, he knows what the bearing should be and he can adjust the horizontal circle of the tacheometer to provide this reading. The instrument is now *oriented* and he can set out (or locate) the EC.

The process is repeated at the BC.

In theory, the surveyor can now set his instrument up at EC 5 and sight back to Beacon B to reorient the tacheometer and then proceed to set out the tangent. This is by having his survey assistant and assistant's assistant move in the appointed direction armed with a hammer, steel tape and several pegs smiting the last-mentioned into the ground with the first mentioned at regular intervals as measured by the middle-mentioned. Alternatively he could send his assistant galloping off to BC 6 with a ranging rod and then use that as a line of sight according to which the tangent can be set out. Neither theory is all that good.

The distance between survey beacon and EC is short, typically less than 100 m, whereas the tangent being set out could be anything up to 10 km long. The accuracy of orientation is such that the closing error at the far end would be substantial. It is thus better to align the tache according to points already defined on the tangent being set out. So it would appear that

survey assistant plus ranging rod at the next BC is the preferred option. In fact it is, but for one tiny problem. The suggestion was made that the tangent could be 10 km long.
The likelihood of the rod actually being visible is remote. After all, the curvature of the Earth suggests that the horizon is about 9 km away ...

It is thus necessary to select intermediate points along the tangent and to repeat the process described in respect of the EC, thereafter setting out the tangent between these intermediate points.

If the surveyor were to set up at the EC and have pegs located at distances which are, say, at multiple of twenty metres from this point he would be treating himself, the designer and the contractor to wholly unnecessary problems. Setting out means placing a peg on line at a prescribed distance, the stake value, *from the starting point.* He must therefore now what the stake value of the EC is.

It is safe to assume that the stake value of PI 1 is going to be selected to be zero. The distance between PI 1 and PI 2 has already been calculated so that the stake value of PI 2 is known. Subtracting the tangent distance of Curve 1, T_1, from this stake value provides the stake value of BC 1. The stake value of the EC is NOT the stake value of PI 2 plus the tangent distance, T_1. The curve length is the appropriate distance to add to the BC stake value to derive that of the EC and this is given as

$$L = R.\text{Ə}_{Rad} = R \,.\, \Theta° / 57{,}29578$$

with the factor allowing for the fact that Θ would otherwise be expressed in radians.

To summarise: $SV_{BC} = SV_{PI} - T$

And $SV_{EC} = SV_{BC} + L$

Knowing what the stake value of the EC is, it is a simple matter to establish the position on the defined tangent of the next full stake value, and pegging proceeds in a routine fashion commencing from this point.

Taping is not without its little problems. Stake values are predicated on horizontal distances and the topography is seldom flat. In consequence, if the taping team religiously puts in pegs at twenty metre intervals, they will inevitably pick up a closing error at the next known point. An experienced tape man will thus add a slope correction to the distance taped. Seeing that the distance between known points should be not more than about 500 m and for preference about 200 m, errors of judgment will not be all that pronounced and, in any event won't be cumulative from end to end of the road. The alternative is to provide the tape man with a clinometer (which is a device now found only in museums or in conjunction with a thermometer and spring balances when precise taping with a calibrated steel tape is being carried out) and a table of slope corrections. In any event, precise taping relies invariably on optical measures using infra-red light or laser or the more old-fashioned Tellurometer (invented in South Africa, by the way). The alternative is simply not worth the effort.

4.2 Setting out curves

The likelihood that the surveyor will ever set out the PI of a curve is very low and for two reasons. It is more than likely that the PI will be inaccessible by being inside a cliff face or out in space or across a river. The second reason is more significant and that is that he doesn't really need it any way. He will however have already located the BC and EC of the curve, and this is all he needs.

A circular curve results from a constant rate of change of heading with increasing arc length. From the geometry of the circle, if the total change in bearing (the deviation angle) across the length of the curve is Θ then the angle between the first bearing and the line of sight between the BC and the EC is $\Theta/2$. The change in bearing per metre length of curve is thus

$$\delta\Theta \quad = \quad \Theta/2L$$

The surveyor thus calculates a set of values of bearing appropriate to the various stake values to be set out.

EXAMPLE:

Curve 16 is to be set out and the following information has been provided:

PI Coords	2 367,52	13 733,89
PI Stake value	2 542,37	
Radius	600,00	
Bearing 1	23^0 10' 15"	
Bearing 2	48^0 32' 30"	
Stake interval	20 m	

Calculation of terminals

Item	BC		EC		Ray
	Y	X	Y	X	
Co-ords of PI	2 367,52	13 733,89	2 367,52	13 733,89	135,054
Difference	-53,14	-124,16	101,22	89,42	203^0 10'15"
Co-ords	2 314,38	13 609,73	2 468,74	13 823,31	48^0 32' 30"

Stake values

PI	2 542,37
Tangent	135,054
BC 16	2 407,316
Curve length	265,683
EC 16	2 672,999

δΘ

$\Theta/2 \quad = \quad (48^0\ 32'\ 30'' - 23^0\ 10'\ 15'')/2$

$\quad = \quad 12^0\ 41'\ 07,5''$

$\delta(\Theta/2) \quad = \quad 0,047\ 746\ ^0/m$

For 20 m	=	0,954 929
For 12,685 m	=	0,605 664
For 12,999 m	=	0,620 656

Intermediate bearings for setting out

Stake value	δΘ	Θ	Bearing
2407,315		23,170 833	23^0 10' 15"
2420	0,605 664	23,776 497	23 46 35
2440	0,954 929	24,731 426	24 43 53
2460		25,686 355	25 41 11
2480		26,641 284	26 38 29
2500		27,596 213	27 35 46
2520		28,551 142	28 33 04
2540		29,506 071	29 30 22
2560		30,461 000	30 27 40
2580		31,415 929	31 24 57
2600		32,370 858	32 22 15
2620		33,325 787	33 19 33
2640		34,280 716	34 16 51
2660		35,235 645	35 14 08
2672,998	0,620 656	35,856 301	35 51 23

Unfortunately, the designer neglected to tell the surveyor that the curve actually went around the side of a hill so that the curve couldn't be set out from the BC - at least not all the way. The surveyor managed to get about half way round before the top of the ranging rod irrevocably disappeared from view. Being equal to the occasion, he moved his tacheometer to SV 2 520 and sighted back to the BC, setting a value of 208^0 33' 04" on the horizontal circle. Swinging back through 180^0, he was once more in business with an oriented tacheometer and could set out SV 2 540 by referring to the figures on his table. Or could he?

He could, if he wasn't particularly interested in his curve actually closing.

In getting to SV 2 520, the deflection angle, $\Theta/2$, became 5^0 22' 49". The tangent to the curve thus rotated through 10^0 45' 38". The true bearing of the tangent is therefore

23^0 10' 15"
10^0 45' 38"
33^0 55' 53"

so that by adding 0^0 57' 18" which is the deflection angle for twenty metres
the true bearing is 34^0 53' 11" between SV 2520 and SV 2540.

However, the surveyor would prefer to cut out recalculation of the balance of the curve and simply use the next value on his table, which is 29^0 30' 32". The difference between the true and the preferred values of bearing is 5^0 22' 49" so that all he has to do is to subtract this value from the bearing used for orientation and he is back in business. All that this exposition says is that, regardless of where he is on the curve, he orients by sighting back to the BC, and sets his horizontal circle as though this direction is the reciprocal of the bearing of the first straight.

Measurement of distance is never actually along a true curve but along a series of chords which are relatively short in relation to the radius of the curve. The error thus made is minuscule, being 0,3 mm over a distance of 20 m in the case of a 1 000 m radius curve and 8 mm for a 200 m radius curve. In the latter case, a purist may wish to tape chord lengths of 19, 992 metres. On the other hand, if the curve happens to be on a gradient of 3 % say the positive slope correction to be added is also 8 mm over 20 metres, thus cancelling out the correction already applied. In the best traditions of swings and roundabouts, most surveyors don't bother.

5 TRIANGULATION

In the calculations discussed above, only a single ray was involved and the available data was sufficient to fix either the distance and bearing between two known points or, given one set of co-ordinates and the bearing and distance to the unknown point, its co-ordinates were uniquely defined. In summary, from four items of information, two unknown items could be calculated. Consider the fact that the standard form of a linear equation is

$$y = ax + b$$

If the two parameters, a and b, of the relationship are known, the single value of y, the dependant variable can be calculated for any value of x. In co-ordinate calculation two items of information have to be calculated, hence the need for two equations and four items of basic input information.

Triangulation is the name given to the process whereby the properties of the triangle are used to calculate the co-ordinates of an unknown point. Triangulation is brought into play when the available data is not so conveniently arranged that either a simple Polar or Join will suffice. For example, the bearing to the unknown point from one known point and the distance from another may be known. The trick is to reorder this information so that the unknown co-ordinates can be calculated.

5.1 Proper identification of a triangle

The very name "triangle" suggests that three angles are involved. It is however possible to have three connected rays, A B and C with B connected to A and to C, which do not form a closed figure. In this case there are only two angles, hence the fact that "triangle" in relation to a closed figure is a better descriptor than "triside", unless the definition explicitly refers to a closed figure.

Traversing a closed figure to end up pointing in the original direction obviously entails a course change of 360^0. Mathematicians express this more elegantly by saying that it is axiomatic that the external angles of a closed figure sum to 360^0.
Considering this in relation to a triangle, each external angle is also

$$\begin{array}{lcl} \text{Ext} < A & = & 180 - \text{Int} < A \\ \text{Ext} < B & = & 180 - \text{Int} < B \\ \text{Ext} < C & = & 180 - \text{Int} < C \\ 360 & & 540 - (\text{Int} < A + < B + < C) \end{array}$$

from which it follows that the sum of the internal angles of a triangle is 180^0.

This bit of pure wisdom makes it possible, in terms of the fundamental linear relationship, to derive one new item of information, to whit the magnitude of the third angle of the triangle, from two items of information, being the magnitude of the other two angles. It is then possible to construct a triangle knowing only what the magnitude of two of the angles is. Unfortunately, the triangle is not uniquely defined, because

- ❑ it could have London, San Franciso and Pretoria as the points of the triangle or the points could be a pinhead away from each other. Some indication of scale of the triangle is required such as its area or the length of one of the sides

- the sequence of the angles hasn't been defined so that, effectively, one could be looking at it from the front or the back.

With these further bits of information, it is possible to uniquely identify and thus construct the triangle.

To the surveyor, the problem is however not adequately resolved because he needs one further item of information and that is the orientation of the triangle. Is the base line of the triangle North-South or anything else through a range of 360^0? Only with this last piece of information also in hand is he in a position to uniquely solve the triangle.

5.2 Solving the triangle

All triangulation problems commence with knowledge of the co-ordinates of two points. This knowledge identifies the scale of the triangle and its orientation, eg the baseline is 5 km long and has a bearing of 135^0 27' 46". The additional information required in order to geometrically construct the triangle could be a combination of

(a) the lengths of the other two sides of the triangle, whereby the position of the third point is fixed by the intersection of two arcs. Of course, the arcs intersect twice so that the surveyor is still required to resolve which of the two available answers is the one that he wants.

(b) the length of one side and the opposite angle. The triangle is then defined by the intersection of a line at the given bearing from the appropriate point on the baseline and an arc around the other end of the base line.

(c) the two angles between the baseline and the unknown point which is then fixed by the intersection of the two rays.

The basic principle is that, if the triangle can be drawn on paper, the position of its points can be calculated. Calculation takes the form of deriving the information needed to be able to ultimately carry out a Join or a Polar from one or other of the two points defining the base line.

(a) Two sides known (additional to the baseline length between A and B)

From the figure it can be seen that the height of the triangle, h, can be expressed as either

$$h = (l_A^2 - x^2)^{0.5}$$

or

$$h = [l_B^2 - (l_{AB} - x)^2]^{0.5}$$

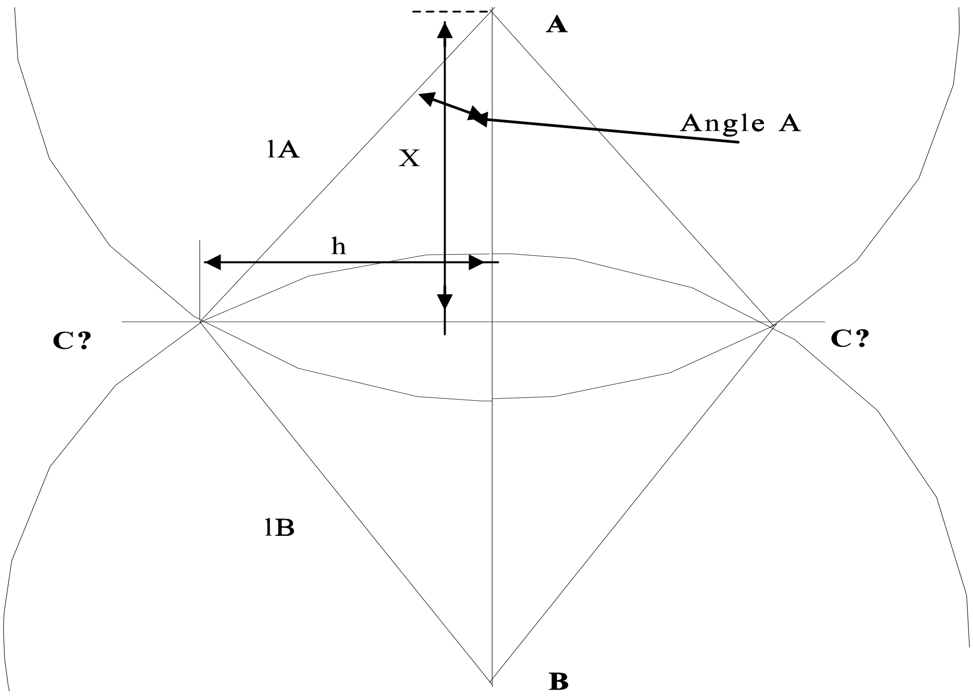
from which it can be deduced that

$$x = \frac{l_B^2 - l_{AB}^2 + l_A^2}{2l_A}$$

and

$$\cos A = x/l_A$$

so that the co-ordinates of Point C can be calculated by a Polar from Point A in terms of the distance l_A and the bearing calculated from the bearing between Points A and B plus or minus the angle A.



(b) the length of one side known and the opposite angle (additional to the baseline length)

In this case the well-known Sine Rule can be brought into play. This states that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Using this relationship, a (which is the side opposite to angle A and hence nothing more or less than l_B) and sin A are known as is c (otherwise known as AB), so that sin C is calculable. With angles A and C known it follows that angle B is known. A polar from B using the bearing from B and the distance l_B gives the co-ordinates of the unknown point.

(c) two angles known

This is also an application of the Sine Rule. The two known angles define the third so that both unknown sides can be calculated. The co-ordinates of the unknown point are once again established using a polar.

6 RESECTION

The above calculations are based on the fact that the surveyor can set up at a point, the co-ordinates of which are known. Resection is the process of deriving the co-ordinates of the set-up point from observations to known points. A case in point is creating a survey beacon close to the route from observations to remote trigonometrical survey beacons.

The process can be geometrically compared to plotting, on a piece of tracing paper, rays from a point with the angles, as observed in the field, between them. The tracing paper is then slid over a plan on which the known points have been plotted until the rays pass through the known points. The point of intersection of the rays is the location of the unknown point.

A minimum of three known stations is required to derive a unique answer so that resection is also known as the three-point problem. If, in addition to the angle at the unknown point, only two points are known, the solution is not unique. The best-known example of this is that if a circle is constructed such that its diameter is equal to the base line length, the angle at any point of the circumference of the circle is a right angle, ie half the central angle which in this special case happens to be 180^0. Any angle at the circumference less than 180^0 simply means that the baseline is a chord of the circle and not the diameter (which, after all, is simply a special case chord).

The figure illustrates the exercise.

Join A -- B provides the length AB and the bearing from A to B
Join B -- C provides the length BC and the bearing from B to C

and

$$\delta = BC - AB + 180$$

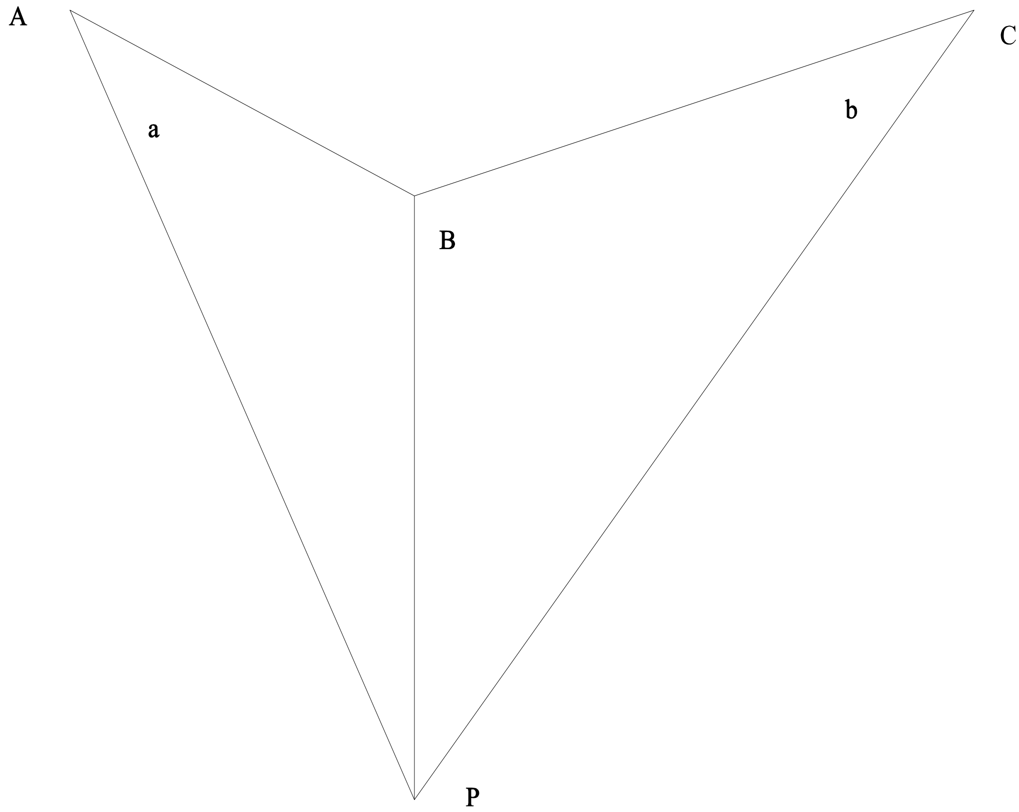
so that

$$a + b = 360 - (\alpha + ß + \delta)$$

The magnitude of angle a is given by

$$\cot a = \frac{AB.\ \sin \beta}{BC\ .\ \sin \alpha.\ \ \sin (a + b)} + \cot (a + b)$$

and with angle a known, angle b is also known.



The Sine Rule gives

$$BP = AB\ .\ \sin a / \sin \alpha$$

and $$BP = BC\ .\ \sin b / \sin \beta$$

which provides calculation and check calculation of the length of BP

With two angles of the triangle ABP known the third is known, similarly the third angle of triangle CBP is known, hence providing calculation and check calculation of the bearing BP.

With the length and bearing of BP known and the co-ordinates of B given, a polar from B to P provides the co-ordinates of the latter.